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1999 J. Phys. A: Math. Gen. 32 5807

(http://iopscience.iop.org/0305-4470/32/31/306)

J. Phys. A: Math. Gen. **32** (1999) 5807–5822. Printed in the UK PII: S0305-4470(99)01493-6

Magnetic monopoles, electric neutrality and the static Maxwell–Dirac equations

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Received 1 February 1999, in final form 25 May 1999

Abstract. We study the full Maxwell–Dirac equations: Dirac field with minimally coupled electromagnetic field and Maxwell field with Dirac current as source. Our particular interest is the static case in which the Dirac current is purely time-like—the 'electron' is at rest in some Lorentz frame. In this case we prove two theorems under rather general assumptions.

Firstly, that if the system is also stationary (time independent in some gauge) and isolated (in the sense that the fields belong to a suitable weighted Sobolev space), then the system as a whole must have vanishing total charge, i.e. it must be electrically neutral. In fact, the theorem only requires that the system be *asymptotically* stationary and static.

Secondly, we show, in the axially symmetric case, that if there are external Coulomb fields then these must necessarily be magnetically charged—all Coulomb external sources are electrically charged magnetic monopoles.

1. Introduction

The Maxwell–Dirac equations are the classical field (or, more 'traditionally', first quantized) equations for electronic matter. Historically, only the linearized equations (where the Dirac current is ignored as a source for the Maxwell equations) have been studied in detail—for a comprehensive survey of the Dirac equation with various potentials, see Thaller [1]. The lack of past interest in the full Maxwell–Dirac equations is partly due to the very difficult nonlinearities of the equations. More importantly, the classical problem was swamped by the extraordinary success of QED.

The difficult nature of these nonlinear equations has meant that the existence theory has only recently been enunciated—some highlights in this development might be Gross [2], Chadam [3], Georgiev [4], Esteban *et al* [5], and Bournaveas [6]. This work culminated in a tour de force of nonlinear functional analysis, the global existence proof of Flato *et al* [7].

Our aim in studying the Maxwell–Dirac system is to look for possible non-linear behaviour which would not be apparent in perturbation expansions. The particular solutions found in [9, 10] exhibit just this sort of behaviour—localization and charge screening. See also Das [8] and the recent work of Finster *et al* [11].

The static Maxwell–Dirac equations were first written down in [9]. In the present work we use this formulation to prove two theorems. Firstly, that the stationary, static Maxwell–Dirac system must have vanishing total charge; this is done in section 4. The second theorem proves that, in the axially symmetric case an external Coulomb field must have an associated

0305-4470/99/315807+16$30.00

magnetic charge—external Coulomb fields must be electrically charged magnetic monopoles. This theorem is proved in section 6.

2. The static Maxwell–Dirac equations

In standard notation the Maxwell–Dirac equations are

$$\begin{aligned}
&\gamma^{\alpha}(\partial_{\alpha} - \mathrm{i}eA_{\alpha})\psi + \mathrm{i}m\psi = 0 \\
&F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta} \\
&\partial^{\alpha}F_{\alpha\beta} = -4\pi e j_{\beta} = -4\pi e\bar{\psi}\gamma_{\beta}\psi.
\end{aligned}$$

In [9] the 2-spinor form of the Dirac equations was employed to solve for the electromagnetic potential, under the non-degeneracy condition $j^{\alpha}j_{\alpha} \neq 0$. Requiring A^{α} to be a real four-vector then gave a set of partial differential equations in the Dirac field alone, the reality conditions.

For 2-spinors u_A and v^B we have

$$\psi = \begin{pmatrix} u_A \\ \bar{v}^{\dot{B}} \end{pmatrix} \qquad \text{with} \quad u_C v^C \neq 0 \quad \text{(non-degeneracy)}.$$

The electromagnetic potential,

$$A^{A\dot{A}} = \frac{\mathrm{i}}{e(u^c v_c)} \left\{ v^A \partial^{B\dot{A}} u_B + u^A \partial^{B\dot{A}} v_B + \frac{\mathrm{i}m}{\sqrt{2}} (u^A \bar{u}^{\dot{A}} + v^A \bar{v}^{\dot{A}}) \right\}.$$

The reality conditions,

$$\begin{aligned}
&\partial^{A\dot{A}}(u_A \bar{u}_{\dot{A}}) = -\frac{\mathrm{i}m}{\sqrt{2}}(u^C v_C - \bar{u}^{\dot{C}} \bar{v}_{\dot{C}}) \\
&\partial^{A\dot{A}}(v_A \bar{v}_{A}) = \frac{\mathrm{i}m}{\sqrt{2}}(u^C v_C - \bar{u}^{\dot{C}} \bar{v}_{\dot{C}}) \\
&u_A \partial^{A\dot{A}} \bar{v}_{\dot{A}} - \bar{v}_{\dot{A}} \partial^{A\dot{A}} u_A = 0.
\end{aligned}$$

The Maxwell equations,

$$\partial^{\alpha}F_{\alpha\beta} = -4\pi e j_{\beta} = -4\pi e \sigma_{\beta}^{A\dot{A}} (u_A \bar{u}_{\dot{A}} + v_A \bar{v}_{\dot{A}}).$$

These equations constitute the Maxwell–Dirac system.

We next impose the static condition.

Definition 1. *The Maxwell–Dirac system is said to be static if there exists a local Lorentz frame in which the Dirac current vector is purely time-like, i.e.* $j^{\alpha} = j^0 \delta_0^{\alpha}$.

Imposing this condition one quickly finds that

$$v^A = \mathrm{e}^{\mathrm{i}\chi} \sqrt{2} \sigma^{0A\dot{A}} \bar{u}_{\dot{A}} \qquad \text{with } \chi \text{ a real function.}$$

The current vector is now

$$j^{\alpha} = \sqrt{2}(u^0 \bar{u}^{\dot{0}} + u^1 \bar{u}^{\dot{1}}) \delta_0^{\alpha}.$$

As noted in [9] the gauge is fixed by the choice,

$$\begin{aligned}
&u^0 = X \mathrm{e}^{\frac{\mathrm{i}}{2}(\chi+\eta)} \\
&u^1 = Y \mathrm{e}^{\frac{\mathrm{i}}{2}(\chi-\eta)}
\end{aligned} \tag{1}$$

with X, Y, and η real functions on $\mathbb{R}^4$.

Defining the null vector L,

$$L = (\sigma^{\alpha}_{A\dot{A}} u^A \bar{u}^{\dot{A}}) = \left(L^0, \frac{1}{\sqrt{2}}\boldsymbol{V}\right) \qquad \text{with} \quad L^0 = \frac{1}{\sqrt{2}}(X^2 + Y^2)$$
$$\boldsymbol{V} = (2XY\cos\eta, 2XY\sin\eta, X^2 - Y^2)$$

our equations become,

$$\frac{\partial}{\partial t}(X^2 + Y^2) = 0$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = -2m(X^2 + Y^2)\sin\chi$$
$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{\nabla}\chi)\times\boldsymbol{V} = \boldsymbol{0}.$$

With electromagnetic potential

$$A^0 = \frac{m}{e}\cos\chi + \frac{(X^2 - Y^2)}{2e(X^2 + Y^2)}\frac{\partial\eta}{\partial t} + \frac{(\boldsymbol{\nabla}\chi)\cdot\boldsymbol{V}}{2e(X^2 + Y^2)}$$
$$\boldsymbol{A} = \frac{1}{2e(X^2 + Y^2)}\left[\frac{\partial\chi}{\partial t}\boldsymbol{V} + (X^2 - Y^2)\boldsymbol{\nabla}\eta - \boldsymbol{\nabla}\times\boldsymbol{V}\right]$$
$$\text{where} \quad \boldsymbol{A} = (A^1, A^2, A^3).$$

The full system is given by the above two sets of equations and the Maxwell equations. Some solutions to the the static Maxwell–Dirac equations were presented in [9, 10].

One further condition we want to impose is that of stationarity.

Definition 2. *The Maxwell–Dirac system will be called stationary if there exists a gauge in which $\psi = e^{i\omega t}\phi$, with the bi-spinor ϕ independent of t. Such a gauge will be referred to as a stationary gauge.*

A stationary gauge is not, of course unique. For a stationary Maxwell–Dirac system there is a distinguished stationary gauge, that for which ψ is indepenendent of t. Clearly, for such a stationary system, if we have $A^0 \to 0$ as $r \to \infty$ in some stationary gauge then $A^0 \to$constant as $r \to \infty$ in any other stationary gauge.

We now have the following simple lemma.

Lemma 1. *The static Maxwell–Dirac system is stationary if and only if, in the gauge given in (1),*

$$\frac{\partial\eta}{\partial t} = 0 \qquad \textit{and} \qquad \frac{\partial X}{\partial t} = 0.$$

In the stationary case we also have,

$$\frac{\partial\chi}{\partial t} = 0 \qquad \textit{and} \qquad \frac{\partial\boldsymbol{V}}{\partial t} = \boldsymbol{0}.$$

Proof. If the system is stationary there exists a gauge transformation such that

$$u^A \to e^{i\xi}u^A = e^{i\omega t}\zeta^A \qquad \text{with} \quad \zeta^A \quad \text{independent of } t.$$

Consequently,

$$Xe^{\frac{i}{2}(\chi+\eta)} = e^{i(\omega t - \xi)}\zeta^0$$
$$Ye^{\frac{i}{2}(\chi-\eta)} = e^{i(\omega t - \xi)}\zeta^1.$$

So, $|X| = |\zeta^0|$ and $|Y| = |\zeta^1|$, both independent of t. We also have,

$$\frac{\partial\chi}{\partial t} + \frac{\partial\eta}{\partial t} = 2\left(\omega - \frac{\partial\xi}{\partial t}\right)$$
$$\frac{\partial\chi}{\partial t} - \frac{\partial\eta}{\partial t} = 2\left(\omega - \frac{\partial\xi}{\partial t}\right).$$

So that $\frac{\partial\eta}{\partial t} = 0$. The argument is easily reversed to get the converse statement. □

3. Isolated systems

An isolated system is one for which all sources are contained in some ball B_ρ ($\rho < \infty$) and for which the fields die off as $|x| = r \to \infty$.

In what follows we will be considering stationary Maxwell–Dirac systems. For such systems we would expect, in an appropriate stationary gauge, that A^α should be $\mathrm{O}(1/r)$ as $r \to \infty$. We will also need to impose some decay conditions on ψ as $r \to \infty$ in order to appropriately define an isolated Maxwell–Dirac system. The best language for the discussion of such decay conditions and other regularity issues is the language of weighted function spaces; specifically weighted classical and Sobolev spaces.

We will use the definitions of [12], other accounts of the theory may be found in [13–15].

Definition 3. *Weighted Sobolev spaces can be defined via the weighted Lebesgue spaces L^p_δ, $1 \leqslant p \leqslant \infty$ which are spaces of locally measurable functions for which the norms*

$$\|f\|_{p,\delta} = \begin{cases} \left(\int_{\mathbb{R}^n} |f|^p \sigma^{-p\delta-n}\,\mathrm{d}x\right)^{\frac{1}{p}} & p < \infty \\ \operatorname{ess\,sup}_{\mathbb{R}^n}(\sigma^{-\delta}|f|) & p = \infty \end{cases}$$

are finite. We can replace $\mathbb{R}^n$ with subsets Ω of $\mathbb{R}^n$ in these definitions. The weight σ is usually taken to be $\sigma = \sqrt{1+r^2}$ or $\sigma = r$ on subsets excluding $\{0\}$. The weighted Sobolev spaces are now defined as consisting of functions with weak derivatives up to order k for which the following norm is finite:

$$\|f\|_{k,p,\delta} = \sum_{j=0}^{k} \|D^j f\|_{p,\delta-j}.$$

For $p < \infty$ we denote the weighted Sobolev space by $W^{k,p}_\delta$. For $p = \infty$ we denote the classical weighted function space by C^k_δ.

We will also require the Sobolev inequality, given here in the form presented in [12].

Sobolev inequality. *If $F \in W^{k,p}_\delta$ then*

(i) $\|f\|_{\frac{np}{(n-kp)},\delta} \leqslant C\|f\|_{k,q,\delta}$, *if* $n - kp > 0$ *and* $p \leqslant q \leqslant \frac{np}{(n-p)}$

(ii) $\|f\|_{\infty,\delta} \leqslant C\|f\|_{k,p,\delta}$, *if* $n - kp < 0$, *and* $|f(x)| = \mathrm{o}(r^\delta)$ *as* $r \to \infty$.

Another tool we will require is the 'multiplication lemma'. In what follows we will be mainly using Sobolev spaces with $p = 2$, and it is for this case that we give the multiplication lemma (adapted from [13]).

Multiplication lemma. *Pointwise multiplication on $\mathbb{R}^n$ is a continuous bilinear mapping*

$$W^{k_1,2}_{\delta_1} \times W^{k_2,2}_{\delta_2} \to W^{k,2}_\delta$$

if $k_1, k_2 \geqslant k$, $k < k_1 + k_2 - n/2$, *and* $\delta > \delta_1 + \delta_2$.

In discussing the decay conditions on ψ it will be useful to have some new notation. Firstly, we introduce two new 2-spinors o_A and ι_A by,

$$\begin{aligned} u_A &= \sqrt{R}\mathrm{e}^{\frac{1}{2}\mathrm{i}\chi} o_A \\ v_A &= \sqrt{R}\mathrm{e}^{\frac{1}{2}\mathrm{i}\chi} \iota_A \\ \text{with} \quad & \iota^C o_C = 1. \end{aligned} \tag{2}$$

Here, R is a positive real function and χ a real function (in fact, as it turns out, the same function introduced in the static case). Our non-degeneracy condition now reads

$$v^A u_A = R\mathrm{e}^{\mathrm{i}\chi} \neq 0.$$

The dyad o_A, ι_A give a spinor dyad basis which is 'co-moving' with the 'Dirac flow' given by j^α. In general we have

$$j^\alpha = R\sigma^\alpha_{A\dot{A}}(o^A\bar{o}^{\dot{A}} + \iota^A\bar{\iota}^{\dot{A}})$$

with $j^\alpha j_\alpha = 2R^2$. These ideas can be developed further and lead to a Newman–Penrose (NP)-type formalism for the Maxwell–Dirac system (for the NP formalism in general relativity see [16]). We will not fully pursue this here, although we go somewhat down this path in our proof of lemma 2, below.

For our static systems we have

$$\begin{aligned} R &= X^2 + Y^2 \\ \boldsymbol{V} &= R\hat{\boldsymbol{V}} = R(\sin\tau\cos\eta, \sin\tau\sin\eta, \cos\tau) \\ X &= \sqrt{R}\cos\frac{\tau}{2} \\ Y &= \sqrt{R}\sin\frac{\tau}{2} \\ (o_A) &= \begin{pmatrix} \sin\frac{\tau}{2}\mathrm{e}^{-\frac{\mathrm{i}}{2}\eta} \\ -\cos\frac{\tau}{2}\mathrm{e}^{\frac{\mathrm{i}}{2}\eta} \end{pmatrix} \\ (\iota^A) &= \begin{pmatrix} \sin\frac{\tau}{2}\mathrm{e}^{\frac{\mathrm{i}}{2}\eta} \\ -\cos\frac{\tau}{2}\mathrm{e}^{-\frac{\mathrm{i}}{2}\eta} \end{pmatrix}. \end{aligned} \tag{3}$$

Notice that o_A and ι^A are both $\mathrm{O}(1)$ as $r \to \infty$, so ψ decays as $\sqrt{R}$.

In discussing decay conditions on the A^α we need, of course to be aware that A^α is defined only up to a gauge transformation. This problem is usually resolved (to some extent) by imposing gauge conditions, such as the Lorenz gauge. We restrict our attention to stationary gauges and demand that A^0 be $\mathrm{O}(1/r)$ as $r \to \infty$ in some gauge. If we demand that $\boldsymbol{A}$ be $\mathrm{O}(1/r)$ then the question is, can we solve

$$\nabla^2\phi + \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$$

so that $\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\phi$ is $\mathrm{O}(1/r)$? That is, can we find a gauge transformation which takes us to the Lorenz gauge with the new $\boldsymbol{A}$ still satisfying the appropriate $\mathrm{O}(1/r)$ decay? The answer is yes provided we choose the original $\boldsymbol{A}$ in the correct function space.

We need A^α to be at least twice (weakly) differentiable to make sense of the Maxwell equations. So to get the appropriate differentiability and decay we will take (see definition, below)

$$A^\alpha \in W^{2,p}_{-1+\epsilon}(E_\rho)$$

for some $p > \frac{3}{2}$ and all $\epsilon > 0$. Here, $E_\rho = \mathbb{R}^3 \backslash B_\rho$, with $\rho < \infty$ large enough so that B_ρ encloses all external sources. This means we can now solve the gauge equation for

$\phi \in W^{3,p}_{\epsilon}(E_\rho)$, with $\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\phi \in W^{2,p}_{-1+\epsilon}(E_\rho)$, in fact, for $0 < \epsilon < 1$ the Laplacian gives an isomorphism between the function spaces $W^{3,p}_{\epsilon}$ and $W^{1,p}_{-2+\epsilon}$, see [12].

The Maxwell equations imply that $j^\alpha \in W^{0,p}_{-3+\epsilon}(E\rho)$. In the vector basis co-moving with j^α (induced from the co-moving dyad) the charge density is $\sqrt{2}R$, we would expect therefore that R is $\mathrm{O}(1/r^3)$ as $r \to \infty$. We also require at least three derivatives for R to define the Maxwell equations when the A^α are written in terms of the components of ψ. This suggests we should take $R \in W^{3,p}_{-3+\epsilon}(E_\rho)$. The o_A and ι_A must also have at least three (weak) derivatives and we also need to ensure that $j^\alpha \in W^{0,p}_{-3+\epsilon}(E_\rho)$. We will require $o_A, \iota_A \in W^{3,p}_{\epsilon}(E_\rho)$, for any $\epsilon > 0$. This leaves the differentiability and decay of χ to be determined. Again we will require at least three derivatives of χ. The decay rate, however, must be determined from the equations.

We can now give our definition of an isolated Maxwell–Dirac system. For concreteness and ease of manipulation we will restrict our attention to Sobolev spaces $W^{k,2}_{\delta}(E_\rho)$.

Definition 4. *A stationary Maxwell–Dirac system will be said to be isolated if, in some stationary gauge, we have*

$$\psi = \mathrm{e}^{\mathrm{i}Et} R \begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}\chi}{2}} o_A \\ \mathrm{e}^{-\frac{\mathrm{i}\chi}{2}} \bar{\iota}_{\dot{A}} \end{pmatrix}$$

with E constant and

$$R \in W^{3,2}_{-3+\epsilon}(E_\rho) \qquad o_A, \iota_A \in W^{3,2}_{\epsilon}(E_\rho) \qquad \textit{and} \qquad A^\alpha \in W^{2,2}_{-1+\epsilon}(E_\rho)$$
$$\textit{for some} \quad \rho > 0 \quad \textit{and any} \quad \epsilon > 0.$$

Remarks.

(a) This definition ensures, after use of the Sobolev inequality and the multiplication lemma, that $\psi = \mathrm{o}(r^{-\frac{3}{2}+\epsilon})$ and $A^\alpha = \mathrm{o}(r^{-1+\epsilon})$.

(b) Note the constant E is fixed in this stationary gauge, for $A^0 \to 0$ as $r \to \infty$. In the distinguished stationary gauge (where all fields are independent of t) we have $A^0 \to \frac{E}{e}$ as $r \to \infty$.

(c) Notice our condition places regularity restrictions on the fields in the region E_ρ only. In the 'interior', $\mathbb{R}^3 \backslash E_\rho$ there are no regularity assumptions. This allows the possibility of external sources such as (singular) Coulomb potentials or magnetic monopoles in this interior region. The total electric charge is still determined by the Gauss integral of the electric field intensity over the 'sphere at infinity'—this will, of course, vanish if A^0 decays faster than $\frac{1}{r}$ as $r \to \infty$.

We are now in a position to prove a lemma which will be used in the proof of theorem 1 in the next section. But we first need some new notation.

We introduce the complex null tetrad vectors

$$l^\alpha = \sigma^\alpha_{A\dot{A}} o^A \bar{o}^{\dot{A}} \qquad n^\alpha = \sigma^\alpha_{A\dot{A}} \iota^A \bar{\iota}^{\dot{A}}$$
$$m^\alpha = \sigma^\alpha_{A\dot{A}} o^A \bar{\iota}^{\dot{A}} \qquad \bar{m}^\alpha = \sigma^\alpha_{A\dot{A}} \iota^A \bar{o}^{\dot{A}}.$$

This null tetrad can now be used to define the following (NP) intrinsic derivatives:

$$D = l^\alpha \frac{\partial}{\partial x^\alpha} \qquad \triangle = n^\alpha \frac{\partial}{\partial x^\alpha}$$
$$\delta = m^\alpha \frac{\partial}{\partial x^\alpha} \qquad \bar{\delta} = \bar{m}^\alpha \frac{\partial}{\partial x^\alpha}.$$

With this notation and the expression for $A^{A\dot{A}}$ of section 2 we find that the (real) potential A^α has to have the following components with respect to the null tetrad:

$$
\begin{aligned}
A_l &= l_\alpha A^\alpha = \frac{1}{2e}[\sqrt{2}m\cos\chi \ - \triangle\chi + \mathrm{i}(\mu - \bar{\mu} + \gamma - \bar{\gamma})] \\
A_n &= n_\alpha A^\alpha = \frac{1}{2e}[\sqrt{2}m\cos\chi + D\chi + \mathrm{i}(\rho - \bar{\rho} + \bar{\varepsilon} - \varepsilon)] \\
A_m &= m_\alpha A^\alpha = \frac{\mathrm{i}}{2e}\left[-\frac{\delta R}{R} + \bar{\alpha} - \beta + \tau - \bar{\pi}\right] \\
A_{\bar{m}} &= \bar{m}_\alpha A^\alpha = \frac{\mathrm{i}}{2e}\left[\frac{\bar{\delta} R}{R} - \alpha + \bar{\beta} - \bar{\tau} + \pi\right].
\end{aligned}
\tag{4}
$$

Here $\alpha, \beta, \tau, \mu, \rho, \gamma$ and ε are the NP spin coefficients (Ricci rotation coefficients for the non-holonomic NP tetrad), see [16]. Their exact form is not important here, what we do need to know is that they are all of the form $o\partial o$, $\iota\partial\iota$, $o\partial\iota$ or $\iota\partial o$, where ∂ is any one of the NP intrinsic derivatives, D, $\triangle$, δ or $\bar{\delta}$.

Lemma 2. *For an isolated, stationary Maxwell–Dirac system the following must hold*

$$
\left.\begin{aligned} \delta R/R - \tfrac{E}{e}m_0 \\ \bar{\delta} R/R - \tfrac{E}{e}\bar{m}_0 \end{aligned}\right\} \in W^{2,2}_{-1+\epsilon}(E_\rho)
$$

for any $\epsilon > 0$ and some constant E.

Proof. A straightforward application of the multiplication lemma. We work in the distinguished stationary gauge. With $o_A, \iota_A \in W^{3,2}_\epsilon(E_\rho)$ and $A^0 - E/e$, $\boldsymbol{A} \in W^{2,2}_{-1+\epsilon}(E_\rho)$, for any $\epsilon > 0$, we have

$$
A_m - m_0\frac{E}{e} \qquad A_{\bar{m}} - \bar{m}_0\frac{E}{e} \in W^{2,2}_{\delta_1}(E_\rho) \qquad \delta_1 > -1 + 3\epsilon
$$

and

$$
\alpha, \beta, \gamma, \tau, \mu, \pi, \varepsilon \in W^{2,2}_{\delta_2}(E_\rho) \qquad \delta_2 > -1 + 4\epsilon.
$$

The result then follows from (4). □

The spherically symmetric solution of [9] is in fact an *isolated*, *stationary* and *static* Maxwell–Dirac system.

4. Vanishing total charge

We will be working with the stationary, static Maxwell–Dirac equations. From section 2 they are

$$
\begin{aligned}
&(\boldsymbol{\nabla}\chi) \times \boldsymbol{V} = \boldsymbol{0} \\
&\boldsymbol{\nabla} \cdot \boldsymbol{V} = -2m(X^2 + Y^2)\sin\chi \\
&A^0 = \frac{m}{e}\cos\chi + \frac{(\boldsymbol{\nabla}\chi)\cdot\boldsymbol{V}}{2e(X^2 + Y^2)} \\
&\boldsymbol{A} = \frac{1}{2e(X^2 + Y^2)}[(X^2 - Y^2)\boldsymbol{\nabla}\eta - \boldsymbol{\nabla} \times \boldsymbol{V}] \\
&\nabla^2 A^0 = 4\pi e j^0 \\
&\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0.
\end{aligned}
\tag{5}
$$

We can now state and prove our theorem of vanishing total charge.

Theorem 1. *An isolated, stationary, static Maxwell–Dirac system is electrically neutral.*

Proof. A more restricted version of this theorem was proved in [17] by one of us (HB).

The stationary gauge of definition 4 is the one for which $A^0 \to 0$ as $r \to 0$, the stationary gauge used in equations (5) has ψ independent of t. A gauge transformation of the type $\psi \to \mathrm{e}^{-\mathrm{i}Et}\psi$ will bring the gauge of definition 4 into that of equations (5). The A^0 of these equations then differs by a constant, $-E/e$, from the A^0 of definition 4. In fact, we will be able to determine E in the proof of the theorem, see also corollary 1. As we noted earlier we have

$$o_A = \begin{pmatrix} \sin\frac{\tau}{2}\mathrm{e}^{-\mathrm{i}\frac{\eta}{2}} \\ -\cos\frac{\tau}{2}\mathrm{e}^{\mathrm{i}\frac{\eta}{2}} \end{pmatrix}$$
$$\iota^A = \begin{pmatrix} \sin\frac{\tau}{2}\mathrm{e}^{\mathrm{i}\frac{\eta}{2}} \\ -\cos\frac{\tau}{2}\mathrm{e}^{-\mathrm{i}\frac{\eta}{2}} \end{pmatrix}$$

in the static case. The system is isolated so using the multiplication lemma we have that

$$\sin\tau, \quad \cos\tau, \quad \sin\eta, \quad \cos\eta \in W^{3,2}_{\epsilon}(E_\rho).$$

Now we have $m_0 = \sigma_0^{A\dot{A}} o_A \bar{\iota}_{\dot{A}} = 0$, so using lemma 2 we have

$$\frac{\delta R}{R} \qquad \text{and} \qquad \frac{\bar{\delta} R}{R} \in W^{2,2}_{-1+\epsilon}(E_\rho).$$

Which in our static case gives,

$$\cos\tau\left(\cos\eta\frac{\partial R}{\partial x} + \sin\eta\frac{\partial R}{\partial y}\right) - \sin\tau\frac{\partial R}{\partial z} \in W^{2,2}_{-1+\epsilon}(E_\rho)$$
$$\text{and} \qquad \sin\eta\frac{\partial R}{\partial x} - \cos\eta\frac{\partial R}{\partial y} \in W^{2,2}_{-1+\epsilon}(E_\rho). \tag{6}$$

The second of equations (5) can be written as

$$-2m\sin\chi = \frac{\boldsymbol{\nabla}\cdot\boldsymbol{V}}{R} = \frac{\hat{\boldsymbol{V}}\cdot\boldsymbol{\nabla}R}{R} + \boldsymbol{\nabla}\cdot\hat{\boldsymbol{V}} \tag{7}$$

with $\hat{\boldsymbol{V}} = (\sin\tau\cos\eta, \sin\eta\sin\tau, \cos\tau)$. Using the multiplication lemma we have

$$\hat{\boldsymbol{V}} \in W^{3,2}_{2\epsilon}(E_\rho) \qquad \text{and} \qquad \boldsymbol{\nabla}\cdot\hat{\boldsymbol{V}} \in W^{2,2}_{-1+2\epsilon}(E_\rho)$$

for any $\epsilon > 0$. We also have

$$\frac{\hat{\boldsymbol{V}}\cdot\boldsymbol{\nabla}R}{R} = \sin\tau\cos\eta\frac{\partial R}{\partial x} + \sin\tau\sin\eta\frac{\partial R}{\partial y} + \cos\tau\frac{\partial R}{\partial z}. \tag{8}$$

Next we utilize the invariance of our equations under Lorentz transformations. In fact, we know that if $(u_A(x^\alpha), v_A(x^\alpha))$ is a solution to the Maxwell–Dirac equations then $(u_A(\hat{x}^\alpha), v_A(\hat{x}^\alpha))$ is a solution to the original system in the x^α coordinates; here $\hat{x}^\alpha$ are the Lorentz transformed Cartesian coordinates. This is, of course, true for any linear Lorentz invariant theory. Consider the rotation

$$\hat{x} = x\cos\omega + y\sin\omega$$
$$\hat{y} = -x\sin\omega + y\cos\omega$$
$$\hat{z} = z \qquad \text{with} \quad \omega = -\frac{\pi}{2}.$$

This gives

$$\frac{\sin\eta(\hat{x}^\alpha)}{R(\hat{x}^\alpha)}\frac{\partial R(\hat{x}^\alpha)}{\partial x} - \frac{\cos\eta(\hat{x}^\alpha)}{R(\hat{x}^\alpha)}\frac{\partial R(\hat{x}^\alpha)}{\partial y} = \frac{\cos\eta(\hat{x}^\alpha)}{R(\hat{x}^\alpha)}\frac{\partial R(\hat{x}^\alpha)}{\partial \hat{x}} + \frac{\sin\eta(\hat{x}^\alpha)}{R(\hat{x}^\alpha)}\frac{\partial R(\hat{x}^\alpha)}{\partial \hat{y}}.$$

Diffeomorphisms $\mathbb{R}^3 \to \mathbb{R}^3$ induce an isomorphism of the Sobolev spaces $W_\delta^{k,p}$, see [13]. The rotation above preserves E_ρ and will give an isomorphism of the Sobolev spaces $W_\delta^{k,p}(E_\rho)$. Using our last equation and (6) gives

$$\frac{\cos\eta}{R}\frac{\partial R}{\partial x} + \frac{\sin\eta}{R}\frac{\partial R}{\partial y} \in W^{2,2}_{-1+\epsilon}(E_\rho).$$

This equation and (6) with the multiplication lemma then gives (multiply each equation in turn by $\sin\eta$ and $\cos\eta$ etc),

$$\frac{1}{R}\frac{\partial R}{\partial x} \qquad \frac{1}{R}\frac{\partial R}{\partial y} \in W^{2,2}_{-1+\epsilon}(E_\rho).$$

The rotation

$$\hat{x} = -z \qquad \hat{y} = y \qquad \hat{z} = x$$

gives

$$\frac{\sin\eta(\hat{x}^\alpha)}{R(\hat{x}^\alpha)}\frac{\partial R(\hat{x}^\alpha)}{\partial x} - \frac{\cos\eta(\hat{x}^\alpha)}{R(\hat{x}^\alpha)}\frac{\partial R(\hat{x}^\alpha)}{\partial y} = \frac{\sin\eta(\hat{x}^\alpha)}{R(\hat{x}^\alpha)}\frac{\partial R(\hat{x}^\alpha)}{\partial \hat{z}} - \frac{\cos\eta(\hat{x}^\alpha)}{R(\hat{x}^\alpha)}\frac{\partial R(\hat{x}^\alpha)}{\partial \hat{y}}.$$

Again using the multiplication lemma with (6), we have

$$\frac{\sin\eta}{R}\frac{\partial R}{\partial z} \in W^{2,2}_{-1+\epsilon}(E_\rho).$$

In the same fashion, using the rotation $\hat{x} = x$, $\hat{y} = z$, $\hat{z} = -y$, we have

$$\frac{\cos\eta}{R}\frac{\partial R}{\partial z} \in W^{2,2}_{-1+\epsilon}(E_\rho).$$

Another use of the multiplication lemma with the last two equations gives $\frac{1}{R}\partial R/\partial z \in W^{2,2}_{-1+\epsilon}(E_\rho)$.

Altogether we have

$$\frac{1}{R}\frac{\partial R}{\partial x} \qquad \frac{1}{R}\frac{\partial R}{\partial y} \qquad \frac{1}{R}\frac{\partial R}{\partial z} \in W^{2,2}_{-1+\epsilon}(E_\rho).$$

A final use of the multiplication lemma with (6) and we have

$$\frac{\hat{\boldsymbol{V}}\cdot\boldsymbol{\nabla}R}{R} \in W^{2,2}_{-1+\epsilon}(E_\rho).$$

We can now conclude from (7) that $\sin\chi \in W^{2,2}_{-1+\epsilon}(E_\rho)$, for any $\epsilon > 0$. By the Sobolev inequality $\sin\chi = \mathrm{o}(r^{-1+\epsilon})$ as $r \to \infty$.

Now sine is an invertible C^∞ function on the range of χ (on E_ρ, with ρ large) with $\sin\chi = 0$ for $\chi = 0$. So we can now write

$$\chi = n\pi + \mu \qquad \text{with} \quad n = 0, \pm1, \pm2, \ldots \qquad \text{and} \qquad \mu \in W^{2,2}_{-1+\epsilon}(E_\rho). \tag{9}$$

Next we use the first of equations (5) to rewrite A^0 entirely in terms of χ and $|\nabla\chi|$. This equation implies we may write $\boldsymbol{V} = \gamma\boldsymbol{\nabla}\chi$, for some function γ. We also have $|\boldsymbol{V}| = X^2 + Y^2 = |\gamma|\,|\nabla\chi|$, so that

$$A^0 = \frac{m}{e}\cos\chi + \frac{\varepsilon}{2e}|\nabla\chi| \qquad \text{with} \quad \varepsilon = \frac{\gamma}{|\gamma|}.$$

Which, using (9), may be written as

$$A^0 - \frac{m}{e}\cos(n\pi) = -\frac{2m}{e}\cos(n\pi)\sin^2\left(\frac{\mu}{2}\right) + \frac{\varepsilon}{2e}|\nabla\mu|.$$

Hence,

$$A^0 - \frac{m}{e}\cos(n\pi) \in W^{1,2}_{-2+\epsilon}(E_\rho).$$

Note that $|\nabla\chi|$ is bounded on E_ρ. So γ cannot change sign on E_ρ as $R = X^2 + Y^2 \neq 0$, ε is fixed.

From the (first) Sobolev inequality applied to $A^0 - \frac{m}{e}\cos n\pi$, with $p = q = 2$, we have that

$$A^0 - \frac{m}{e}\cos(n\pi) \in W^{0,6}_{-2+\epsilon}(E_\rho).$$

Consequently, we also have

$$A^0 - \frac{m}{e}\cos(n\pi) \in W^{0,6}_{-\frac{5}{4}}(E_\rho).$$

Now, in the static case, $j^0 = \sqrt{2}R$, so $\nabla^2(A^0 - \cos n\pi) \in W^{3,2}_{-3+\epsilon}$. Hence,

$$A^0 - \frac{m}{e}\cos(n\pi) \in W^{5,2}_{-1+\epsilon}(E_\rho)$$

and from the Sobolev inequality we find that $|\partial_i A^0| < Cr^{-2+\epsilon}$ (with $\partial_i = \partial/\partial x^i$). So we have,

$$|\partial_i A^0|^6 r^{-6(-1-\frac{5}{4})-3} < C^4 |\partial_i A^0|^2 r^{-2(-3+\epsilon)-3}$$

for any $0 < \epsilon < \frac{1}{12}$. Thus, $\partial_i A^0 \in W^{0,6}_{-9/4}(E_\rho)$. Which finally gives us,

$$A^0 - \frac{m}{e}\cos(n\pi) \in W^{1,6}_{-\frac{5}{4}}(E_\rho)$$

and the Sobolev inequality now gives

$$A^0 - \cos n\pi = \mathrm{o}(r^{-\frac{5}{4}}).$$

From which it is clear that the total electric charge of the system

$$\lim_{\rho\to\infty} \frac{1}{4\pi}\int_{S_\rho} (\boldsymbol{\nabla} A^0)\cdot \mathrm{d}\boldsymbol{S} \qquad \text{with } S_\rho \text{ the sphere of radius } \rho$$

must vanish. □

Corollary 1. *In the gauge in which $A^0 \to 0$ as $r \to \infty$ the Dirac bi-spinor of an isolated, stationary, static Maxwell–Dirac system takes the form*

$$\psi = \mathrm{e}^{\pm \mathrm{i}mt}\phi \qquad \textit{with} \quad \phi \in W^{3,2}_{-\frac{3}{2}+\epsilon}(E_\rho) \quad \textit{(as above)}.$$

Proof. This result is a simple consequence of the proof of the theorem. We had

$$A^0 - \frac{m}{e}\cos(n\pi) \in W^{1,2}_{-2+\epsilon}(E_\rho).$$

The constant term, $-\frac{m}{e}\cos n\pi$ is removed by the gauge transformation

$$\psi \to \mathrm{e}^{-\mathrm{i}\cos(n\pi)mt}\psi.$$

□

In [9] we presented a unique spherically symmetric solution which provides a good example of the properties just described. For large r we found

$$\chi = \pi - \frac{1}{mr} - \frac{1}{168m^3r^3} + \mathrm{O}\left(\frac{1}{r^6}\right)$$

$$A^0 = -\frac{m}{e} + \frac{1}{emr^2} - \frac{3}{112em^3r^4} + \mathrm{O}\left(\frac{1}{r^6}\right).$$

5. The axially symmetric case

By axially symmetric we mean that the system is invariant under rotations about a fixed axis. This requires that gauge invariant quantities be invariant under translations in the azimuthal coordinate ϕ.

Definition 5. *A Maxwell–Dirac system will be called axially symmetric if*

$$\left[L, \frac{\partial}{\partial\phi}\right] = \left[N, \frac{\partial}{\partial\phi}\right] = 0 \qquad \textit{where} \quad L = \sigma^{\alpha}_{A\dot{A}} u^A \bar{u}^{\dot{A}} \frac{\partial}{\partial x^\alpha}$$

$$\textit{and} \qquad N = \sigma^{\alpha}_{A\dot{A}} v^A \bar{v}^{\dot{A}} \frac{\partial}{\partial x^\alpha}.$$

For our static systems we require only that L be invariant under translations in ϕ. In fact, writing L in cylindrical polar coordinates,

$$\begin{aligned} L &= L^0 \frac{\partial}{\partial t} + \frac{1}{\sqrt{2}} (V^\rho \hat{\boldsymbol{\rho}} + V^\phi \hat{\boldsymbol{\phi}} + V^z \hat{\boldsymbol{k}}) \\ &= L^0 \frac{\partial}{\partial t} + \frac{1}{\sqrt{2}} \left(V^\rho \frac{\partial}{\partial\rho} + V^\phi \frac{\partial}{\partial\phi} + V^z \frac{\partial}{\partial z} \right) \end{aligned}$$

we find that

$$L^0 = \frac{X^2 + Y^2}{\sqrt{2}} \qquad V^\rho = 2XY \cos(\eta - \phi)$$

$$V^\phi = 2XY \sin(\eta - \phi) \qquad \text{and} \qquad V^z = X^2 - Y^2$$

must all be independent of ϕ. This means our static Maxwell–Dirac system is axially symmetric if

$$\frac{\partial X}{\partial\phi} = \frac{\partial Y}{\partial\phi} = \frac{\partial(\eta - \phi)}{\partial\phi} = 0.$$

This information lets us characterize stationary, axially symmetric, static Maxwell–Dirac systems as follows.

Lemma 3. *A non-trivial static, axially symmetric Maxwell–Dirac system is stationary if and only if $\eta = \phi$ in the gauge given in (1).*

Proof. As $\boldsymbol{V}$ is independent of ϕ then so is χ, as $\sin\chi = (\boldsymbol{\nabla} \cdot \boldsymbol{V})/(X^2 + Y^2)$. The reality condition,

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{\nabla}\chi) \times \boldsymbol{V} = \boldsymbol{0}$$

gives,

$$\begin{aligned} &\frac{\partial V^\rho}{\partial t} - V^\phi \frac{\partial\chi}{\partial z} = 0 \\ &\frac{\partial V^\phi}{\partial t} - V^z \frac{\partial\chi}{\partial\rho} + V^\rho \frac{\partial\chi}{\partial z} = 0 \\ &\frac{\partial V^z}{\partial t} + V^\phi \frac{\partial\chi}{\partial\rho} = 0. \end{aligned}$$

If the system is stationary, lemma 1 says $\boldsymbol{V}$ is independent of t. So either $V^\phi = 0$ or χ is constant. Constant χ leads to the trivial solution with $X = Y = 0$. So we must take $V^\phi = 2XY \sin(\eta - \phi) = 0$. We have $\eta = \phi \bmod n\pi$.

On the other hand, if $\eta = \phi$ we have $V^\phi = 0$ and consequently both V^ρ and V^z are independent of t. Combining these with $\partial_t(X^2 + Y^2) = 0$ gives the result. $\square$

From now on we study the axially symmetric, stationary, static Maxwell–Dirac equations.

It will prove convenient in what follows to use spherical polar coordinates (r, θ, ϕ) and to make the following change of variables:

$$X = \sqrt{R}\cos\left(\frac{\tau}{2}\right) \qquad \text{and} \qquad Y = \sqrt{R}\sin\left(\frac{\tau}{2}\right).$$

All our dependent variables depend only on (r, θ), the equations are

$$\begin{aligned}
&\boldsymbol{V} = R[\cos(\tau-\theta)\hat{\boldsymbol{r}} + \sin(\tau-\theta)\hat{\boldsymbol{\theta}}] \\
&(\boldsymbol{\nabla}\chi) \times \boldsymbol{V} = \boldsymbol{0} \\
&\boldsymbol{\nabla}\cdot\boldsymbol{V} = -2mR\sin\chi \\
&A^0 = \frac{m}{e}\cos\chi + \frac{\varepsilon}{2e}|\nabla\chi| \\
&\boldsymbol{A} = \frac{1}{2e}\left\{\frac{\cos\tau}{r\sin\theta} - \frac{1}{rR}\left[\frac{\partial}{\partial r}(rR\sin(\tau-\theta)) - \frac{\partial}{\partial\theta}(R\cos(\tau-\theta))\right]\right\}\hat{\boldsymbol{\phi}}
\end{aligned} \tag{10}$$

together with the Maxwell equations. Here $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are the unit coordinate vectors.

We note that A^α automatically satisfies the Lorenz gauge condition.

In the spherically symmetric case [9] we have $\tau = \theta$ with R and χ functions of r only. In which case

$$\boldsymbol{A} = \frac{1}{2e}\frac{\cot\theta}{r}\hat{\boldsymbol{\phi}}$$

the magnetic monopole.

Other tractable cases are those for which τ is constant. It is straightforward to show there are really only two cases, see [17],

- the cylindrically symmetric case, $\tau = \pi/2$, see [10].
- the case $\tau = 0$, variables depend on z only, see [18].

6. Magnetic monopoles

The spherically symmetric solution has an external (i.e. not sourced by the Dirac field directly) electrically charged magnetic monopole. In this section we will prove a theorem which shows that this is, to some extent, the generic situation. We will show that the axially symmetric, stationary, static Maxwell–Dirac system can have an external Coulomb point charge only if it is magnetically charged. First we define what we mean by an external Coulomb field.

Definition 6. *We will say that a Maxwell–Dirac system has an external Coulomb field if we can choose spherical polar coordinates and a ball B_ρ centred at $r = 0$ such that*

$$A^0 = \frac{q}{r} + h \qquad \text{in} \quad B_\rho \quad \rho > 0$$

with, h, a bounded function on B_ρ and q constant.

Remarks.

(1) q/r is, of course, harmonic on $B_\rho\backslash\{0\}$, so it is not directly sourced by the Dirac field via the Maxwell equation $\nabla^2 A^0 = 4\pi e j^0$. In this sense the Coulomb field is 'external' to the Dirac field.

(2) The singular point $r = 0$ of A^0 leads to singular behaviour in $\boldsymbol{A}$ and ψ. We can, however, still define the electric and magnetic charges of a region (or indeed a point, using a suitable limit) by the usual Gauss integrals,

$$\int_S \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{S} \qquad \text{and} \qquad \int_S \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S}$$

where S is a topological sphere enclosing the region for which we wish to calculate the charge.

(3) The condition that h is bounded on B_ρ is quite weak. In practice it will follow from elliptic regularity of the Poisson equation $\nabla^2 h = 4\pi e j^0$ ($j^0 = \sqrt{2}(X^2+Y^2) = \sqrt{2}R$ must be at least $L^1(B_\rho)$ for the total Dirac charge to be well defined on B_ρ), see for example [19,20]. We also require R to be differentiable at least three times (if only in the weak sense) to satisfy the Maxwell equation for $\boldsymbol{A}$—this puts R in $W^{3,p}_\delta$ for some $p \geqslant 1$. Consequently, h will be in $W^{5,p}_{\delta+2}$, which ensures that h can be included in one of the classical weighted function spaces.

Now for our theorem.

Theorem 2. *Suppose an axially symmetric, stationary, static Maxwell–Dirac system has an external Coulomb field. Let* $A_{(\rho,\rho_1)} = B_\rho \backslash B_{\rho_1}$, *with* $\rho > \rho_1 > 0$ *and* B_ρ *as in the definition above. If* $r\partial_r\tau - \partial_\theta R/R$ *is bounded on* $A_{(\rho,\rho_1)}$, *then the Coulomb point charge necessarily carries a magnetic charge of Dirac value,* $\frac{\pm 1}{2e}$; *i.e. all electric point charges also carry a magnetic monopole.*

Remark. The condition $r\partial_r\tau - \partial_\theta R/R$ bounded on $A_{(\rho,\rho_1)}$ is not particularly strong. It is true, for example, if $X, Y \in C^1_0(A_{(\rho,\rho_1)})$—we assume $R = X^2 + Y^2 \neq 0$; or, when $\boldsymbol{A} \in W^{2,2}_1(B_\rho\backslash\{0\})$.

Proof. In [17] it was shown that if a Maxwell–Dirac system has a central Coulomb charge then the magnetic field is necessarily unbounded at the centre.

In our proof of theorem 1 we noted that $\boldsymbol{V} = \gamma\boldsymbol{\nabla}\chi$, so from (10) we have

$$\left.\begin{aligned}\cos(\tau-\theta) &= \varepsilon\tfrac{\partial_r\chi}{|\nabla\chi|}\\ \sin(\tau-\theta) &= \varepsilon\tfrac{\partial_\theta\chi}{r|\nabla\chi|}\end{aligned}\right\} \qquad \text{where} \quad \varepsilon = \frac{\gamma}{|\gamma|}. \tag{11}$$

We will first show that $\sin(\tau-\theta)\to 0$ as $r \to 0$.

Now, $A^0 = \frac{q}{r} + h$, so from (10) we have

$$|\nabla\chi| = \frac{q}{r} + g \tag{12}$$

where $g = h - \frac{m}{e}\cos\chi$ is bounded on B_ρ. Next, we write

$$\chi = q\ln r + \zeta \qquad \text{on} \quad B_\rho\backslash\{0\}.$$

Equation (12) is

$$(\partial_r\zeta)^2 + \frac{2q}{r}\partial_r\zeta + \frac{1}{r^2}(\partial_\theta\zeta)^2 = \frac{2q}{r}g + g^2. \tag{13}$$

Note that $r|\nabla\chi| \to q$ as $r \to 0$ so that $r\partial_r\chi = q + r\partial_r\zeta$ and $\partial_\theta\chi = \partial_\theta\zeta$ are bounded on $A_{(\rho,\rho_1)}$, for ρ small enough. Let η be the smallest non-negative number such that $r^\eta\partial_r\zeta \to 0$ as $r \to 0$. We can write (13) as

$$(r^{\frac{1+\eta}{2}}\partial_r\zeta)^2 + 2qr^\eta\partial_r\zeta + (r^{\frac{\eta-1}{2}}\partial_\theta\zeta)^2 = 2qr^\eta g + r^{\eta+1}g^2.$$

It is clear that $r^{\frac{1+\eta}{2}}\partial_r\zeta \to 0$ and $r^{\frac{\eta-1}{2}}\partial_\theta\zeta \to 0$. From which we conclude that $\eta \leqslant 1$ and so $\partial_\theta\chi = \partial_\theta\zeta \to 0$ as $r \to 0$. Consequently,

$$\sin(\tau-\theta) = \frac{\partial_\theta\chi}{r|\nabla\chi|} \to 0 \qquad \text{as} \quad r \to 0.$$

Now $r\partial_r\chi \to q$ as $r \to 0$, so $r\partial_r\chi$ cannot change sign as θ varies, for ρ small. Writing $\varepsilon_1 = q/|q|$ we have, from (11),

$$\left.\begin{aligned} \cos\tau(r,\pi) &\to -\varepsilon\varepsilon_1 \\ \cos\tau(r,0) &\to \varepsilon\varepsilon_1 \end{aligned}\right\} \qquad \text{as} \quad r \to 0. \tag{14}$$

We now work on $A_{(\rho,\rho_1)}$ with ρ small enough that $\cos(\tau-\theta) \neq 0$. The magnetic charge of the magnetic field $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ in B_r, $\rho > r > \rho_1$, is

$$b = \frac{1}{4\pi}\int_{S_r}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = \frac{1}{2}\int_{\theta=0}^{\pi}\partial_\theta(r\sin\theta A)\,\mathrm{d}\theta \tag{15}$$

where $\boldsymbol{A} = A\hat{\boldsymbol{\phi}}$ is given in (10). After some manipulation the third equation of (10) can be written as

$$\partial_\theta\tau - 2 + \frac{r\partial_r R}{R} = \tan(\tau-\theta)\left(r\partial_r\tau - \frac{\partial_\theta R}{R}\right) - 2mr\sin\chi - 3\cos(\tau-\theta).$$

So that if $r\partial_r\tau - \frac{\partial_\theta R}{R}$ is bounded on $A_{(\rho,\rho_1)}$ then so is $\partial_\theta\tau - 2 + \frac{r\partial_r R}{R}$. From (10) we have

$$\begin{aligned} &r\sin\theta A = \frac{1}{2e}\{\cos\tau - P\sin\theta\} \\ &\text{where} \quad P = \left(r\partial_r\tau - \frac{\partial_\theta R}{R}\right)\cos(\tau-\theta) - \left(\partial_\theta\tau - 2 + \frac{r\partial_r R}{R}\right)\sin(\tau-\theta). \end{aligned}$$

Clearly, under the conditions of the theorem, P is bounded on $A_{(\rho,\rho_1)}$. From (15) we now have

$$\begin{aligned} b &= \frac{1}{2e}\left[\frac{\cos\tau(r,\pi) - \cos\tau(r,0)}{2}\right] - [P\sin\theta]_{\theta=0}^{\pi} \\ &= \frac{1}{2e}\left[\frac{\cos\tau(r,\pi) - \cos\tau(r,0)}{2}\right]. \end{aligned}$$

Finally, from equations (14) and (15) we obtain the magnetic charge in the limit $r \to 0$

$$b_0 = \frac{-\varepsilon\varepsilon_1}{2e} = \frac{\pm 1}{2e}.$$

□

Corollary 2. *Suppose we have an axially symmetric, isolated, stationary, static Maxwell–Dirac system with the only external sources being N isolated electrically charged magnetic monopoles. Let the conditions of theorem 2 apply in the N balls B_{ρ_i}, containing the charges. Let the conditions of theorem 1 apply on $E_{\bar{\rho}}$. Then, if N is even there are $N/2$ positive charges and $N/2$ negative charges (with corresponding monopoles), with the total magnetic charge of the system being zero. If N is odd there are $(N-1)/2$ charges with one sign and $(N+1)/2$ charges with the opposite sign, with the total magnetic charge of the system being $\pm 1/(2e)$.*

Proof. We have N charged monopoles each in a ball B_{ρ_i}, $i = 1, 2, 3 \ldots N$, there are no other external sources and all the B_{ρ_i} are properly contained in $B_{\bar{\rho}}$.

We have $\boldsymbol{A}$ is $\mathrm{O}(1/r)$ so that P, as defined in the above proof, is bounded on $E_{\bar{\rho}})$. Using the results in the proof of theorem 2 and the divergence theorem we find the total magnetic charge of the system.

$$b_{\text{total}} = \lim_{r\to\infty} \frac{1}{4\pi} \int_{S_r} (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{2e} \lim_{r\to\infty} \left[\frac{\cos\tau(r,\pi) - \cos\tau(r,0)}{2} \right]$$
$$= -\frac{\varepsilon}{2e} \sum_{i=1}^{N} \varepsilon_i .$$

This gives,

$$1 \geqslant \varepsilon \sum_{i=1}^{N} \varepsilon_i \geqslant -1$$

from which the results of the corollary follow. □

7. Conclusions

At first sight theorem 1 may not appear all that startling—a system of charges and Dirac fields with a non-zero net charge cannot be in equilibrium, we would not expect it to be static. In classical physics, however, one expects stationary or static systems to be the end point of some time evolution. This clearly cannot be the case for a single isolated electron modelled by the Maxwell–Dirac system. To construct such a model we will have to abandon one, or both, of the static and stationary assumptions.

Theorem 1 is remarkable in the following way: no matter what arrangement of external electric and magnetic fields inside the ball B_ρ, no matter what we do to the topology in B_ρ the total electric charge of the system must be zero. The total charge vanishes purely as a result of the asymptotic decay and regularity conditions.

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